


Tensor Network Contractions for #SAT

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Abstract The computational cost of counting the number of solutions satisfying a Boolean formula, which is a problem instance of #SAT, has proven subtle to quantify. Even when finding individual satisfying solutions is computationally easy (e.g. 2-SAT, which is in P), determining the number of solutions can be #P-hard. Recently, computational methods simulating quantum systems experienced advancements due to the development of tensor network algorithms and associated quantum physics-inspired techniques. By these methods, we give an algorithm using an axiomatic tensor contraction language for n -variable #SAT instances with complexity $O((g + cd)^{O(1)}2^c)$ where c is the number of COPY-tensors, g is the number of gates, and d is the maximal degree of any COPY-tensor. Thus, n -variable counting problems can be solved efficiently when their tensor network expression has at most $O(\log n)$ COPY-tensors and polynomial fan-out. This framework also admits an intuitive proof of a variant of the Tovey conjecture (the r,1-SAT instance of the Dubois–Tovey theorem). This study increases the theory, expressiveness and application of tensor based algorithmic tools and provides an alternative insight on these problems which have a long history in statistical physics and computer science.

Keywords Statistical physics · Complexity · Computational physics · Quantum physics

1 Introduction

Given any property used as a measure of complexity, two natural questions arise: (i) how does this property affect the computation of the problem and (ii) how easy is it to determine the value of said property. In this paper, through the lens of tensor networks we discuss several such properties and address these two questions with regard to counting K-SAT problems. These properties include

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- (1) Tensor networks with the structure of a tree,
- (2) The fan-out of the variables,
- (3) Treewidth, and
- (4) Rényi entropy of tensor network states representing problem instances.

We measure the fan-out of the variables by the number of COPY-tensors in the tensor network. We then describe an algorithm whose time complexity depends on the number of COPY-tensors. While related, this number is not in general equal to treewidth (Proposition 23). We also discuss the Rényi entropy as a measure of complexity, especially with regard to the fact that it may not be defined or easily computed.

1.1 Background

Satisfiability problems and counting problems have several physical analogues and have had a long and fruitful history intersecting physics [1, 2]. In 1982 Barahona proved that finding the lowest energy state of Ising spins acting on a 2D lattice is NP-hard [1]. Other seminal results include the concepts of algorithmic simulated annealing [2] which has long been applied to K-SAT and related problems.

The K-SAT problem on n variables is formed from a conjunction of c clauses. Each clause is the disjunction of K literals (variables appearing in negated or un-negated form). Several celebrated results have shown that the ratio c/n is an indicator of what appears to be an algorithmic phase transition [3].

Related studies have also unveiled indicators of such ‘critical behavior’ in other satisfiability problems [4, 5]. We discuss how the algorithm suggested in Theorem 18 contrasted when compared with treewidth based algorithms [6–10], especially in the critical case when the clause to variable ratio is close to one.

Further subtleties arise when we consider the counting versions of these problems (#2-SAT, #3-SAT, or #SAT in general). This version of the problem asks for the number of satisfying bitstrings of a given function f . For example, although 2-SAT is in P, #2-SAT is #P-complete [11].

Tensor network methods are a collection of techniques to model and reason about multilinear maps. These methods form the backbone of tensor network contraction algorithms to model quantum systems [12–15] and are used in the abstract tensor languages [16] to represent channels, maps, states and processes appearing in quantum theory [17–20].

There is body of recent work which has applied tensor network methods appearing in physics to solve combinatorial problems appearing in computer science [20–26]. Contracting a network to determine the norm of a *Boolean tensor network state* is equivalent to counting the number of satisfying solutions of a Boolean formula. Thus in general, tensor contraction is #P-hard [27].

Boolean tensor network states have appeared in several recent studies. This includes the recent identification of several undecidable problems in tensor network states [24–26, 28]. The use of Boolean states have also suggested the use of entropy as a measure of computational complexity [29] as well as several other interesting applications [30–34].

Relationships between multilinear maps and counting problems have long been known in computer science and physics [16, 35]. Several alternative approaches to the counting problem based on tensor network methods have recently appeared [22, 34, 36] as well as the work done by the present authors [21, 23]. Any problem in #P can be expressed as a #SAT problem, which we express as a tensor network in a standard way [21]. In this paper, we exploit the expression of these problems as multilinear maps to give an algorithm for solving #SAT problems based on the fan-out of the variables. If the fan-out is small, the algorithm

is polynomial. We argue that this is a different measure than treewidth, to which it appears similar.

1.2 Results

Our contribution includes the discussion of tensor network properties and their effect on the computational complexity of tensor contraction. We use the fan-out to classify the complexity of an algorithm we propose. We discuss the use of Rényi entropy as a measure of complexity. We also prove several other results related to counting by tensor contraction. Theorem 13 gives an alternative proof for the r , 1-SAT case of the Tovey–Dubois theorem, proving that all read-once formulas are satisfiable. We propose a tensor contraction algorithm based on the number of COPY-tensors in the network and in Theorem 18 give the running time. This leads to a class of efficiently solvable #SAT instances, outlined in Corollary 20. We compare this to other algorithms based on graph theoretic measures, predominantly those defined in terms of treewidth. We also discuss the computational difficulty of determining if Rényi entropy is defined in Theorems 24 and 26.

1.3 Structure

In Sect. 2 we recall the basic ideas connecting counting, satisfiability, and related physical problems. We then cast these problems into the language of tensor network contraction. In Sect. 3, we prove satisfiability of read-once formulas and then in Sect. 4, we present our tensor contraction algorithm and compare it to other known algorithms. Lastly, in Sect. 5, we discuss how the computational complexity relates to the Rényi entropies of a Boolean state concluding with a discussion in Sect. 6. In Appendix 1 we have included the tensor network definitions required for the proofs appearing in this paper.

2 Counting by Tensor Contraction

We first recall how satisfiability and counting are connected to familiar concepts in physics. Further, determining properties of many physical systems can be rephrased in terms of satisfiability and counting problems.

Each n -variable 3-SAT problem can be associated to a $2^n \times 2^n$ Hamiltonian H indexed by variable assignments. Let

$$\psi_f := \sum_{\mathbf{x}} f(\mathbf{x})|\mathbf{x}\rangle$$

and

$$\psi_f^\perp := \sum_{\mathbf{x}} (1 - f(\mathbf{x}))|\mathbf{x}\rangle = \sum_{\mathbf{x}} \bar{f}(\mathbf{x})|\mathbf{x}\rangle = |+\rangle^{\otimes n} - \psi_f,$$

where $|+\rangle := |0\rangle + |1\rangle$ and $\langle \psi_f | \psi_f^\perp \rangle = 0$. Then for $H := |\psi_f^\perp\rangle\langle \psi_f^\perp|$, (note that for physical considerations, we implicitly assume that f is satisfiable or otherwise H will be the zero matrix) we have that the eigenvectors of H correspond to non-satisfying bitstrings.

Such observations have been central to understanding the computational complexity of determining physical properties of many naturally occurring spin lattice problems [1]. A typical problem includes determining a ground state wave function of an Ising spin system (such as Kirpatrick et al.’s simulated annealing [2]). This approach also leads to the definition of the QMA complexity class and the local Hamiltonian problem, which is QMA-complete [37].

Then we note

$$\min_{\psi} \langle \psi | H(\psi) \rangle \geq 0$$

where ψ is a bit string, and equality implies that ψ encodes a satisfying assignment to f . The ground space of H is spanned by all satisfying bitstrings, which is degenerate when there are multiple satisfying assignments. The canonical partition function

$$Z(H, \beta) := e^{-\beta H}$$

at the $\beta \rightarrow \infty$ limit (corresponding to the temperature $T \rightarrow 0$) in general has a degenerate ground state. Therefore the number of satisfying assignments of f (which we denote as $\#f$) satisfies the following bound:

$$0 \leq \#f \leq \text{Tr } Z(H, \beta).$$

Under this embedding the upper-bound holds with equality in the low-temperature limit of the partition function, i.e.

$$\text{Tr } \lim_{\beta \rightarrow \infty} Z(H, \beta) = \#f. \tag{1}$$

This can then be translated into a tensor contraction problem. Let

$$N := 2^n = \#f + \#\bar{f}$$

which can be alternatively expressed as

$$N = \langle + |^{\otimes n} | \psi_f \rangle + \langle + |^{\otimes n} | \psi_{\bar{f}} \rangle.$$

Then evaluating the limit in (1) is equivalent to evaluating the quantity $\langle + |^{\otimes n} | \psi_f \rangle$, which we show has desirable properties when expressed as a tensor contraction.

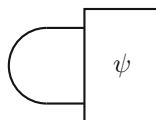
As ψ_f can be viewed as the state of a physical system, we can consider how properties of tensor networks arise from motivations in physics. Penrose related the value of a tensor network contraction to its physical realizeability:

Theorem 1 (Penrose [38]) *The norm of a spin network vanishes if and only if the physical situation it represents is forbidden by the rules of quantum mechanics.*

Example 2 (Example of Penrose’s theorem) Consider a Bell state $\Phi^+ = |00\rangle + |11\rangle$. The amplitude of the first party measuring $|0\rangle$ followed by the second party measuring $|1\rangle$ is zero. This vanishing tensor network contraction is given by $\langle 01 | \Phi^+ \rangle$.

Given a state $|\psi\rangle$, formed by a network of connected tensors, if $\langle \psi | \psi \rangle$ vanishes, the network necessarily represents a non-physical quantum state by Penrose’s theorem. In particular, the following example is a famous instance of non-physicality.

Example 3 (Tensor networks and the grandfather paradox, [25]) Consider the tensor $\psi = |01\rangle + |10\rangle$ contracted in the following network:



The value of this network contraction is

$$(\langle 00 | + \langle 11 |)(|01\rangle + |10\rangle) = \langle 00 | 01 \rangle + \langle 00 | 10 \rangle + \langle 11 | 01 \rangle + \langle 11 | 10 \rangle = 0$$

This is the Grandfather paradox: if you go back in time to kill your Grandfather, you could not have been born to do the deed. Since the value of this tensor network is zero, Theorem 1 tells us that this is indeed a paradox and cannot be realized physically.

2.1 Rephrasing #SAT as a Tensor Network

We describe a way to write a Boolean satisfiability problem as a tensor network, as in [21, 39]. Suppose we are given a SAT formula f and we wish to express it as a tensor ψ_f . Let x_1, \dots, x_n be the variables appearing in this formula. We further suppose that f is given to us in the following form.

Definition 4 A Boolean formula is in *conjunctive normal form* (or CNF form for short) if it is of the form

$$\bigwedge_{i=1}^{\alpha} \left(l_{i,1} \vee \dots \vee l_{i,k_i} \right)$$

where each literal $l_{i,j}$ is of the form x_k or $\neg x_k$. A problem is in r -CNF form if each clause has r literals.

We assign an open wire to each variable x_i . If a variable x_i appears in k clauses, we make k copies of x_i via the COPY-tensor $|0\rangle\langle 0|^{\otimes k} + |1\rangle\langle 1|^{\otimes k}$. We call k the *degree* of the COPY-tensor. It is depicted as a solid black dot in a tensor network.

A Boolean gate (or clause) $\varphi : \{0, 1\}^m \rightarrow \{0, 1\}$ is expressed as the tensor

$$\sum_{\mathbf{x} \in \{0,1\}^m} |\mathbf{x}\rangle\langle \varphi(\mathbf{x})|.$$

Then the COPY-tensor associated to each variable is connected to each tensor representing a clause which that variable appears in. This gives a tensor network with n dangling wires corresponding to the variables and α dangling wires corresponding to the output of each clause. Since we want each clause to have the value of 1, we attach to each of these wires the tensor $|1\rangle$. This is equivalent to composing with the tensor $|1_{\alpha}\rangle = |1\rangle^{\otimes \alpha}$. It is not hard to check that the entire network can be contracted to the tensor

$$\psi_f = \sum_{\mathbf{x}} |\mathbf{x}\rangle\langle f(\mathbf{x})|1_{\alpha}\rangle = \sum_{\mathbf{x}} f(\mathbf{x})|\mathbf{x}\rangle.$$

We call a state of the form ψ_f a *Boolean state*.

Remark 5 (Counting SAT solutions) Let f be a SAT instance. Then $\|\psi_f\|^2$, using the standard norm, counts the number of satisfying solutions of f .

We calculate the inner product of this state with itself viz

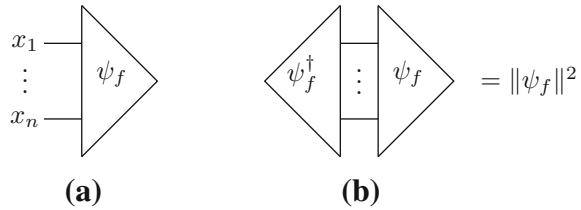
$$\|\psi_f\|^2 = \sum_{\mathbf{x}, \mathbf{y}} f(\mathbf{x})f(\mathbf{y})\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{\mathbf{x}} f(\mathbf{x}), \tag{2}$$

which gives the number of satisfying inputs. This follows since $\langle \mathbf{x}, \mathbf{y} \rangle = \delta_{\mathbf{x}\mathbf{y}}$.

Alternatively, we can count the number of solutions to a Boolean formula f in the following way. For an n -variable Boolean formula f , let $\chi_n = (|0\rangle + |1\rangle)^{\otimes n}$. Then

$$\langle \chi_n, \psi_f \rangle = \sum_{\mathbf{x} \in \{0,1\}^n} f(\mathbf{x})\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{\mathbf{x} \in \{0,1\}^n} f(\mathbf{x}).$$

Fig. 1 Tensor networks that count the number of satisfying solutions for a Boolean formula; **a** an abstract depiction of a Boolean state, and **b** a Boolean state contracted with itself



We note that solving the counting problem for general formula is known to be #P-complete [11]. The condition

$$||\psi_f|| > 0 \tag{3}$$

implies that the SAT instance f has a satisfying assignment. Determining if this condition holds for general Boolean states is an NP-complete decision problem.

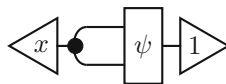
The tensor network contraction for the counting problem is depicted in Fig. 1: (a) gives a network realization of the function and (b) is the contraction that represents the norm of ψ_f . If the value is greater than zero, the instance has a solution.

3 Satisfiability of Certain SAT Instances

We give a few sufficient conditions for satisfiability. The first arises in Sect. 2, namely if the Boolean state represents a physical phenomenon.

Corollary 6 *SAT instances are solvable if and only if the corresponding contraction of a Boolean state with itself (as in Fig. 1b) represents a physical situation allowed by the rules of quantum mechanics Theorem 1.*

Example 7 What does it mean that a SAT formula cannot be realized physically? Consider Example 3 above. We demonstrate how to view this as a SAT formula. We first put a variable on the wire. Then we change ψ to $|01\rangle\langle 1| + |10\rangle\langle 1|$. Then we attach an output to ψ and set it equal to 1. The picture is



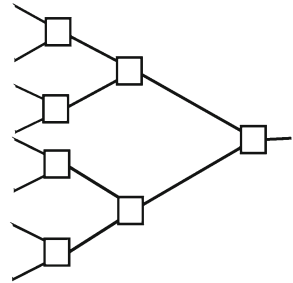
Since x can be zero or one, it is represented by the tensor $\langle 0| + \langle 1|$. The dot is the COPY-tensor. Then we contract

$$\begin{aligned} & (\langle 0| + \langle 1|)(|0\rangle\langle 00| + |1\rangle\langle 11|)(|01\rangle\langle 1| + |10\rangle\langle 1|)(|1\rangle) \\ & = (\langle 00| + \langle 11|)(|01\rangle + |10\rangle) = 0. \end{aligned}$$

We see that this collapses to the grandfather paradox in Example 3. But now we can interpret this as a SAT formula. Here ψ corresponds to the formula $x \wedge \neg x$.

Another class of SAT instances are always satisfiable simply by the structure of their tensor networks, namely trees where every variable is a leaf. These are more familiarly known as read-once functions. Here, we do not demand that the tensor network representation of the be in the form described in Sect. 2.1. Instead, we look at tensor networks as depicted in Fig. 2, where the variables are the left-side leaves, the right-side leaf is the output of the function, and each box is any Boolean function except the two constant functions. Such a network can always be rewritten into CNF form.

Fig. 2 A Boolean function expressible as a tree



Definition 8 (*read-once*) A function f is called *read-once* if it can be represented as a Boolean expression using the operations conjunction, disjunction and negation in which every variable appears exactly once. We call such a factored expression a read-once expression for f .

We call such a function ROF for short. The definition implies that there are no COPY-tensors since every variable can only appear in one of the Boolean tensors in the network. As such, it does indeed have a tree structure when expressed as a tensor network.

These formulas represent a special subclass of r, s -SAT, which is defined to be the decision problem for SAT formulas written in r -CNF form where each variable appears in at most s clauses. Read once formulas, given in CNF form, represent the cases where r is general and $s = 1$. These problems have been studied classically and we mention two celebrated results:

Theorem 9 (Tovey [40]) *Every instance of r, r -SAT is satisfiable.*

Theorem 10 (DuBois [41]) *If every instance of r_0, s_0 -SAT is satisfiable, then r, s -SAT is satisfiable for $r = r_0 + \lambda$ and $s \leq s_0 + \lambda \lceil s_0/r_0 \rceil$, $\lambda \in \mathbb{N}$.*

This implies that r, s -SAT is satisfiable for $s \leq r$ by Theorem 9 and letting $\lambda = 0$ in Theorem 10. In particular, all ROF are satisfiable. Using tensor networks, we can give a very short proof of this.

Definition 11 Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function. Considering the tensor $\tilde{\psi}_f = \sum_{\mathbf{x} \in \{0,1\}^n} |f(\mathbf{x})\rangle \langle f(\mathbf{x})|$, we call a tensor of the form $\tilde{\psi}_f^\dagger \tilde{\psi}_f$ a *diagonal map*.

Lemma 12 *For a Boolean function f , $\tilde{\psi}_f^\dagger \tilde{\psi}_f = \#f^{-1}(0)|0\rangle\langle 0| + \#f^{-1}(1)|1\rangle\langle 1|$ where $\#f^{-1}(b)$ denotes the size of the pre-image of b . Furthermore, if f is not a constant function, we can normalize it to get a tensor ζ_f such that $\zeta_f^\dagger \zeta_f = \text{id}$.*

Proof

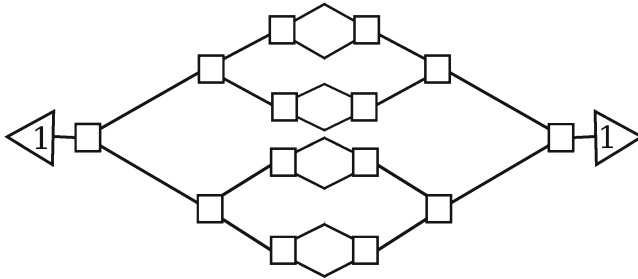
$$\begin{aligned} \tilde{\psi}_f^\dagger \tilde{\psi}_f &= \sum_{\mathbf{x}, \mathbf{y}} |f(\mathbf{x})\rangle \langle \mathbf{x} | \mathbf{y} \rangle \langle f(\mathbf{y})| \\ &= \sum_{\mathbf{x}} |f(\mathbf{x})\rangle \langle f(\mathbf{x})| \\ &= \#f^{-1}(0)|0\rangle\langle 0| + \#f^{-1}(1)|1\rangle\langle 1|. \end{aligned}$$

Now let $\zeta_f = \sum_{\mathbf{x}} \sqrt{\#f^{-1}(f(\mathbf{x}))}^{-1} |f(\mathbf{x})\rangle \langle f(\mathbf{x})|$. It is clear that ζ_f has the desired property and is well defined since we assumed that f was non-constant. □

For a Boolean state representing an ROF, we replace every gate with its normalization. This gives positive weights to the different assignments of variables. The contraction then sums up the weights of the satisfying assignments. It is clear that normalizing a ROF does not change the fact that the norm will be zero if and only if it has no satisfying solution. The resulting scalar, however, will no longer reflect the number of satisfying solutions.

Theorem 13 *Every ROF built up from non-constant gates is satisfiable.*

Proof To show that f is satisfiable, we must show that $\|\tilde{\psi}_f|1\rangle\|^2 \geq 1$. This tensor network is depicted in the picture below.



We normalize every gate $\tilde{\psi}_g$ in the tree using Lemma 12. As discussed above, this new tensor network will have norm zero if and only if it is unsatisfiable. We observe a series of nested diagonal maps. As any ROF does not allow a constant Boolean gate, by Lemma 12 these maps are equivalent to the identity. So we can successively collapse them until we get the contraction $\langle 1|(id)|1\rangle = 1$. So it is indeed satisfiable. \square

4 Efficient Tensor Contraction Algorithm for High Correspondence Networks

Consider a tensor network with a known efficient, i.e. polynomial, contraction method. By inserting new wires and tensors into the network, one can construct a tensor network which is not necessarily easy to contract. Of interest are ways to reduce tensor networks to those that are computationally tractable in an efficient way.

In particular, we are interested in transforming Boolean states into others with polynomial time evaluations. One approach is to restrict the number of tensors added to be at most $O(\log n)$, where n is the number of variables.

Definition 14 (*Tensor network correspondence*) Let v be the minimum number of vertices needed to be removed from the network T , with their incident edges becoming dangling wires, to transform it into a network U known to be efficiently contractible. The *correspondence* of T with respect to U is defined to be $1/v$.

Example 15 (*Low correspondence*) It is well known that stabilizer circuits are efficiently contractible. A polynomial number of stabilizer gates together with the addition of polynomially many single qubits gates, such as a phase gate, results in the stabilizer class of circuits being elevated to a universal set of quantum gates. In general, these networks are as difficult to contract as any problem in QMA, and the correspondence of this family of networks is inversely proportional to some polynomial in the total number of single-qubit gates.

For our purposes, we will be interested in restricting our definition of correspondence to the number of COPY-tensors needed to be removed to recover a tensor network that can be quickly evaluated. In this way, we can construct a wide class of counting problems that can be solved efficiently using tensor contraction algorithms. Here we focus on the Boolean states that are close to tree tensor networks.

It is well known that tree tensor networks are polynomially contractible in the number of gates [8,42,43]. For Boolean states, the gate count includes the tensors $\langle 0| + \langle 1|$ placed on the variable wires.

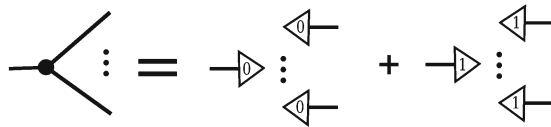
Definition 16 Given a rooted tree, a *limb* is a sequence of vertices v_1, \dots, v_k such that v_2, \dots, v_{k-1} each only have one child and v_{i+1} is a child of v_i .

Proposition 17 An n -variable Boolean state that is a tree has $O(n)$ gates.

Proof A Boolean state has a natural root: the output bit. We argue that the limbs of a Boolean state are of length at most three. Take a limb v_1, \dots, v_k . Then consider $v_i, i \neq 1, k$. It is a unary operation on clauses, implying that it must either be the identity or negation operator. We can therefore assume that there is at most a single negation operator between v_1 and v_k and nothing else. So at worst the tree is a perfect binary tree which has $O(n)$ gates. \square

So we can conclude that the norm of a Boolean state that is a tree can be computed in polynomial time in the number of variables.

Let X be a Boolean state. Corollary 29 (in Appendix 1) allows us to remove a COPY-tensor, resulting in a sum of tensor networks:



We note that there are two summands, where either 0 or 1 is attached to the dangling wires resulting from the removal of the COPY-tensor. Given a tensor network X , if we remove a COPY-tensor, we denote the two summands X_0 and X_1 . Furthermore, we have that

$$C\{X, X\} = \sum_i C\{X_i, X_i\}.$$

We can now give an algorithm for contracting a tensor network, assuming we have an algorithm for contracting a tree tensor network in polynomial time. Let C_1, \dots, C_m be the COPY-tensors appearing in X . Then

$$C\{X, X\} = \sum_{i_1, \dots, i_m} C\{X_{i_1, \dots, i_m}, X_{i_1, \dots, i_m}\}$$

where X_{i_1, \dots, i_m} is the tree formed by removing all the COPY-tensors and assigning the value i_k to the wires that were incident to C_k . So the algorithm for computing $C\{X, X\}$ computes the expression as a sum of contractions of trees.

Theorem 18 (Upper bounding tensor contraction in terms of COPY-tensors) *Given a tensor network as described in Sect. 2.1, the complexity of evaluating this network is $O((g + cd)^{O(1)}2^c)$ where c is the number of COPY-tensors, g is the number of gates, and d is the maximal degree of any COPY-tensor.*

Proof Contracting a tensor network is upper bounded by evaluating the expression

$$\mathcal{C}\{X, X\} = \sum_{i_1, \dots, i_m} \mathcal{C}\{X_{i_1, \dots, i_m}, X_{i_1, \dots, i_m}\}.$$

For each COPY-tensor removed, we double the number of summands. We also add a gate to each dangling wire created by removing a COPY-tensor. So the computation is bounded by summing over the contraction of 2^c trees, each of which can be contracted in time $O((g + cd)^{O(1)})$ [8]. \square

Remark 19 We must mention that Theorem 18 only applies when the Boolean state is given by a tensor network as described in Sect. 2.1. Otherwise, there are many tensor networks to describe a given Boolean state, many of which invalidate the statement of Theorem 18.

Corollary 20 (A class of efficiently solvable #SAT instances) *Suppose that a Boolean state with n variables has inverse logarithmic correspondence with a tree tensor network and the maximal degree of the COPY-tensors is $O(n^\alpha)$. Then this Boolean state can be contracted in polynomial time.*

Proof Inverse logarithmic correspondence implies that if the SAT instance has n variables, then it has $O(\log(n))$ COPY-tensors. By Theorem 18, the complexity of contraction is $O((g + n^\alpha \log(n))^{O(1)} 2^{\log(n)})$. By Proposition 17, g is $O(n)$, so the algorithm presented will contract the tensor network in polynomial time. \square

Theorems 9 and 10 only pertain to instances of r, s -SAT that are always satisfiable, they do not address the complexity of the corresponding counting problem. For instance, read-twice monotone 3-CNF formulas are always satisfiable by Theorems 9 and 10, but counting the number of solutions is #P-hard [44].

Our results, in contrast, give conditions for the tractability of the counting problem associated to r, s -SAT. One of these conditions is not implicit in the definition of r, s -SAT, namely that we bound the number of variables that can appear in more than one clause. To restate Corollary 20, we have shown that # $r, \text{poly}(n)$ -SAT is polynomial time countable if the number of COPY-tensors is bounded by $\log(n)$.

4.1 Relation to Other Algorithms

There have been several results relating the complexity of contracting tree tensor networks to various measures such as treewidth. An overview can be found in [6]. The general type of result is that the time to solve a #SAT instance is polynomial in the number of variables and exponential in some notion of width [7, 10].

We do wish to briefly discuss how our algorithm compares with algorithms based on treewidth of the Boolean state and the relationship between treewidth and the number of COPY-tensors. Contracting a tree tensor network T takes time $g^{O(1)} \exp(\text{tw}(T))$, where $\text{tw}(T)$ is the treewidth of T [8]. More explicitly, we have the following results:

Theorem 21 (Fisher et al. [7]) *Given an n -variable SAT formula with treewidth k , there is an algorithm counting the number of solutions in time $4^k (n + n^2 \log_2(n))$.*

Theorem 22 (Samer and Szeider [9]) *Given a SAT formula with treewidth k , largest clause of size l , and N the number of nodes of the tree-decomposition, the #SAT problem can be solved in time $O(4^k l N)$.*

Proposition 23 *Given a Boolean state, let k be the treewidth and c the number of COPY-tensors. Then $k \leq c$.*

Proof It is well known that if H is a subgraph of G , then $\text{tw}(H) \leq \text{tw}(G)$. If we place the tensor $\langle 0| + \langle 1|$ on each variable wire and then compose with the COPY-tensors to get a tensor network T , a variable will correspond to a leaf if and only if it appears in exactly one clause. The treewidth of T with n variables, n' of which are not leaves, and m clauses, is at most the treewidth of $K_{n',m}$, which is known to be $\min(m, n')$. This is because adding leaves to a graph does not increase the treewidth. On the other hand, $c = n'$. \square

It is easy to show that this inequality can be strict, and that c is not in general equal to cliquewidth or branchwidth either. Note that if the number of variables is large with respect to the number of clauses, then c may be much larger than the treewidth, k . However, the algorithms are comparable as long as c is bounded by 2^k by Theorems 18 and 21. We note that if the clause to variable ratio is extremely small or large, the treewidth is small and the tensor network can be contracted efficiently. In the critical case when $c \approx k$, our algorithm runs *exponentially faster* in k . There is the added advantage that, unlike treewidth, calculating c is not NP-complete. As such, c is an attractive estimate for the complexity of the counting problem. However, the tradeoff is that it is a cruder measurement.

5 Rényi Entropy and Complexity

Rényi entropies [45] have recently been proposed as an indicator of the complexity of counting problems [29]. The Rényi entropies are attractive as they can be viewed as providing a basis to express entanglement monotones [46] and relate to the free-energy of a physical system [47]. We look at the question pertaining to the use of Rényi entropies as a measurement of counting complexity. As we are interested in properties of tensor networks as measures of complexity, a natural question is the efficiency of determining the Rényi entropy of a given state. Many proposed measures of complexity are difficult to determine. As we noted in Sect. 4, the number of COPY-tensors can be easily computed, whereas treewidth cannot. Another issue arises from the fact that Rényi entropies are defined for physical states, which by Theorem 1 does not include non-satisfiable instances of #SAT.

In Sect. 2, we described how to assign a Hamiltonian to a Boolean state. We choose a partition for the rows and columns into two disjoint subsystems, A and B . This is called a *bipartition* of the Hamiltonian, denoted $A : B$. The Rényi entropy of order q with respect to a bipartition $A : B$ is defined to be

$$H_q^{AB}(\rho) = \frac{1}{1-q} \ln \text{Tr}\{\rho^q\} = \frac{1}{1-q} \ln \sum_i \lambda_i^q,$$

$$\rho = \frac{1}{Z_0} \text{Tr}_A\{|\psi\rangle\langle\psi|\}.$$

Here, $Z_0 = \text{Tr}\{|\psi\rangle\langle\psi|\}$. The case $q \rightarrow 0$ gives the rank of the bipartition and the positive sided limit $q \rightarrow 1$ recovers the familiar von Neumann entropy $H_{q \rightarrow 1}^{AB}(\rho) = -\text{Tr}\{\rho \ln \rho\}$.

In terms of the counting problem, note that the state $|\psi\rangle$ could be given by the zero vector, in which case it is not possible to define the state ρ : so more generally we can consider the unnormalized variant of ρ , which is defined as $\rho = \text{Tr}_A\{|\psi\rangle\langle\psi|\}$. This has the following computational significance.

Theorem 24 (Rényi entropy implies satisfiability) *For any bipartition, deciding if the Rényi entropy is defined is NP-hard.*

Proof Given an unnormalized Boolean state ρ ,

$$\rho = \text{Tr}_A\{|\psi\rangle\langle\psi|\}$$

for some bipartition $A : B$, $H_q^{AB}(\rho)$ takes a finite value if and only if the Boolean state is satisfiable. Note that this is independent of choice of bipartition as $\ln \text{Tr}\{\rho^q\}$ is undefined if $|\psi\rangle = 0$. Thus determining if the Rényi entropy is defined is NP-hard. \square

We can then consider instead the promise problem called Unambiguous-SAT: Given a SAT instance promised to have at most one solution, determine if it is satisfiable. Unambiguous-SAT is still a hard problem.

Theorem 25 (Valiant-Vazirani [48]) *If there is a polynomial time algorithm for solving Unambiguous-SAT, then $\text{NP} = \text{RP}$.*

Theorem 26 (Rényi Entropy Reduction to Unambiguous-SAT) *Suppose you have a Boolean state with the promise that the Rényi entropies are all undefined or zero. Then deciding if the Rényi entropies are undefined or zero is as hard as Unambiguous-SAT.*

Proof The Rényi entropies are zero if and only if the density operator is a product state, which implies that the corresponding Boolean state has a unique solution. It is undefined if and only if it has no solution as previously discussed. Therefore, if we are promised the Rényi entropies are zero or undefined for all partitions, we are promised that there is at most one solution. This is precisely Unambiguous-SAT. \square

What we see is that while Rényi entropies may encode information about the complexity of a counting problem, they are unfortunately not easy to compute. Furthermore, they may not even be defined and determining this is computationally challenging.

6 Discussion

The tensor network approach offers several straightforward proofs related to counting problems—including a simple proof of a the $r,1$ instance of the Dubois–Tovey theorem. In addition to this, a core result of this study was presenting a tensor contraction algorithm for n -variable #SAT instances with complexity $O(n^{O(1)2^c})$, where c is the number of COPY-tensors in the network, each with polynomial degree. This provides a class of counting problem instances which can be solved efficiently when their tensor network expression has at most $O(\log c)$ COPY-tensors.

One can view this work as a continuation of the ideas of classically simulating a quantum algorithm [21] to solve a given computational task, which might provide methods which are both quick to develop and scale favorably. Our results were all inspired and conducted in the language of tensor networks.

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Appendix 1: Properties of Tensor Contraction

Although the graphical manipulation of tensors (e.g. algebraic rewrite rules) has been axiomatized by several communities, basic theorems and techniques of contraction are less systematically described. Networks have seen little effort toward their systematic formulation. Hence we present the minimal properties of tensor contraction we relied on to provide the main results in the paper.

We assume that the reader has some basic familiarity with tensor networks and thus focus on the properties of contraction properties which we used. This section should hopefully be self contained. For additional background information, see e.g. [46].

The *value* of a tensor network is obtained by performing all possible partial contractions diagrammed by the network (the order of contraction does not affect the value). If we allow one or more of the tensors in the network to vary, and the network has no dangling wires, we obtain a multilinear function from the direct sum over vertices of the spaces $\bigotimes_{e \in \mathcal{N}(v)} V(e)$ to the complex numbers. When there are dangling wires, the multilinear function's codomain is the tensor product of the vector spaces corresponding to the dangling edges.

We denote such a function by \mathcal{C} when the tensor network defining it is understood. Then multilinearity can be rephrased as follows.

Remark 27 (Linearity of tensor contraction) Tensor contraction is linear in its component tensors provided each appears only once: $\mathcal{C}\{\alpha A_{v_1}, \dots, \alpha A_{v_k} + A'_{v_k}, \dots, A_{v_n}\} = \alpha \mathcal{C}\{A_{v_1}, \dots, A_{v_k}, \dots, A_{v_n}\} + \mathcal{C}\{A_{v_1}, \dots, A'_{v_k}, \dots, A_{v_n}\}$, and then $A \mapsto kA$ we readily find that the contraction becomes $\mathcal{C}\{A'\} + \mathcal{C}\{B\}$ and $k \cdot \mathcal{C}\{A\}$ respectively.

Several of our results rely on transforming contracted sub-networks, into a sum-over products, and vice versa. In other words, removing a contracted tensor (or part of a contracted tensor) and replacing it by a sum of contracted tensors, where each term contains contractions that are multiplied together. The stage for this can be set by stating several properties of tensor contractions. We will use these properties to reshape networks, such that their transformed geometry is known to be efficiently contractible.

Theorem 28 (COPY-tensors as a resolution of identity) *The following sequence of graphical rewrites hold.*

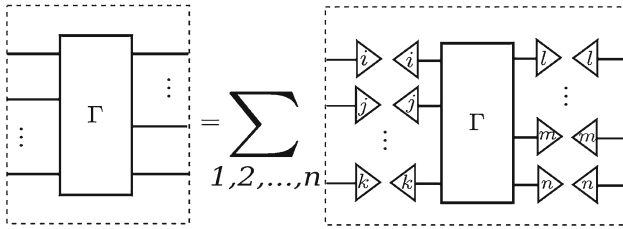
$$\text{---} = \text{---} \begin{array}{c} \bullet \\ \downarrow \\ \nabla \\ \text{+} \end{array} = \sum \text{---} \begin{array}{c} \bullet \\ \downarrow \\ \nabla \\ \text{+} \end{array} = \sum \text{---} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

Proof In the above figure, on the left, we abstractly depict a otherwise arbitrary tensor network, by showing only one single wire. The unit for the COPY-tensor is the plus state $|+\rangle = \sum |n\rangle$. Contracting $|+\rangle$ against the COPY-tensor therefore gives $\sum_{i,n} |i\rangle \langle i| \langle n| = \sum_n |n\rangle \langle n|$ which is precisely the identity. The last equality follows directly from the linearity of tensor contraction. □

This procedure can be iterated over many wires. In particular, if we have a gate in a tensor network, we can remove it from the network by the above method resulting in a nested sum of tensor networks. Of course the number of tensor networks in the summand grows exponentially in the number of wires of the gate removed.

Some counting algorithms are only polynomial time on planar graphs, and this cutting procedure can be used to deal with handles, yielding a running time factor exponential in the genus of the graph [49].

Corollary 29 (Contraction to sum of products transform) *Given a tensor $\Gamma^{...ijk}_{lmn...}$ in a fully contracted network (e.g. a network without open legs), the following graphical identities, which are equal, transforms the contraction, to a sum over products.*



The comprising rectangle above is meant as an abstraction depicting a fully contracted but otherwise general tensor network.

As mentioned, the contraction to sum of products transform in Theorem 29 as well as the tensor network resolution of identity in Theorem 28 will be useful to transform networks into those known to be efficiently contractible.

Theorem 30 (The Cauchy–Schwarz inequality) *Given a contracted network and a partition into two halves x , and y . Writing the contraction as $C\{x, y\}$ the following inequality holds.*

$$C\{x, y\}^2 \leq C\{x, x\} \cdot C\{y, y\}$$

with graphical depiction.



Proof By the linearity of tensor contraction, we arrive at an abstract form of the Cauchy–Schwarz inequality, with equality in the contraction if and only if $x = \alpha \cdot y$. This leads directly to the concept of an angle between tensors,

$$\cos \theta_{xy} = \frac{C\{x, y\}^2}{C\{x, x\} \cdot C\{y, y\}}$$

where the right side is either real valued, or we take the modulus. □

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