Classification of time-symmetry breaking in quantum walks

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Use of quantum walks

Quantum walks on graphs represent an established model capturing essential physics behind a host of natural and synthetic phenomena:

• Proven to provide a universal model of quantum computation
• Capture the core underlying physics of several biological processes in which quantum effects play a role
• A ‘single particle quantum walker’ moves on a graph, with dynamics governed by Schrodinger’s equation and quantum theory predicts the probability of a walker to transition between the graphs’ nodes
• Any quantum process in finite dimensions can be viewed as a single particle quantum walk—so the model is general in this regard
Continuous time quantum walks

A continuous time quantum walk is a quantum operator \( U(t) = e^{-iHt} \) acting on a normalized vector

- for quantum search the all-ones vector
- for transport, a state in the site-basis

The Hamiltonian \( H \) is an adjacency matrix of a weighted bidirected graph: the Hermitian constraint being that \( H = H^\dagger \)

- The quantum evolution of \( U(t) \) can be seen as inducing a walk on the graph the support of \( H \) describes
- This sets a stage to employ techniques from graph and network theory
Quantum walks on complex networks

- **Classification of time-symmetry breaking in quantum walks**
  with Jacob Turner
  Draft available on request. (2016)

- **Chiral Quantum Walks**
  with Dawei Lu and others

- **Quantum Transport Enhancement by Time-Reversal Symmetry Breaking**
  with Zoltan Zimboras, Mauro Faccin, Zoltan Kadar, and James Whitfield
  Scientific Reports 3, 2361 (2013)
Why time-symmetry breaking?

Until recently, quantum walks implicitly modeled only probability transition rates between nodes which were symmetric under time inversion

• Breaking this symmetry provides an arena to consider applications of this symmetry breaking and to better understand its foundations

• The main application discovered so far is that this symmetry breaking can be utilized as a passive means to control and direct quantum transport
How time-symmetry breaking?

The corresponding graph of $H$ is typically considered as being symmetric, with real valued (and possibly even negative edge weights): but it needn’t be.

- More precisely, an edge $e$ connecting vertices $v$ and $w$ can have conjugate weights with respect to $H$ depending if it is viewed as an edge from $v \to w$ or visa versa.
Quantum walk Hamiltonian

- In the standard literature on continuous time quantum walks, the time-independent walk Hamiltonian is defined by a real weighted adjacency matrix $J$ of an underlying undirected graph,

$$H_{QW} = \sum_{n,m} J_{nm} (|n\rangle\langle m| + |m\rangle\langle n|)$$  \hspace{1cm} (1)

The condition that the hopping weights $J_{nm}$ are real numbers implies that the induced transitions between two sites are symmetric under time inversion.
Chiral quantum walk Hamiltonian

We can break this symmetry while maintaining the hermitian property of the operator by appending a complex phase to an edge: $J_{nm} \rightarrow J_{nm} e^{i\theta_{nm}}$
resulting in a continuous time chiral quantum walk (CQW) governed by

$$H_{CQW} = \sum_{n,m} J_{nm} e^{i\theta_{nm}} |n\rangle \langle m| + J_{nm} e^{-i\theta_{nm}} |m\rangle \langle n|$$  \hspace{1cm} (2)
Open systems framework

We investigated this with respect to coherent quantum dynamics and incoherent dynamics within the Markov approximation. Both types of evolutions are included in the Lindblad equation

$$\frac{d}{dt} \rho(t) = \mathcal{L}\{\rho\} = -i[H_{CQW}, \rho]$$

$$+ \sum_k L_k \rho L_k^\dagger - \frac{1}{2} \left( L_k^\dagger L_k \rho + \rho L_k^\dagger L_k \right)$$

where $\rho(t)$ is the density operator describing the state of the system at time $t$ and $L_k$ are Lindblad operators inducing stochastic jumps between quantum states.
Definition of time-symmetry

$P$ is a probability function, $s$ and $f$ are basis elements representing nodes in a graph. A process is time-symmetry iff $\forall t$

$$ P_{s\rightarrow f}(t) - P_{f\rightarrow s}(t) = 0 $$

(4)

- In the case of quantum, it follows that the above is equivalent to

$$ P_{s\rightarrow f}(t) - P_{s\rightarrow f}(-t) = 0 $$
1 Introduction
   Collaborators

2 Toy versions

3 Quantum version
Collaborators

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Jacob Turner  Seth Lloyd  Tomi Johnson

Classification of time-symmetry breaking in quantum walks
Stochastic toy version

Let $S$ be a valid infinitesimal stochastic generator of a continuous-time Markov process, so $s_{ij} \geq 0$ for $i \neq j$, and $\sum_j q_{ij} = 0$ for all $i$. Then $U(t) = e^{tS}$ is a stochastic semi-group. We define the probability function $P_{s \rightarrow f} = U(t)_{fs} = \langle f | U | s \rangle$. The ‘stochastic probability current’ is defined by $\hat{s}_{s,f}(U(t)) := U(t)_{fs} - U(t)_{sf} = 0$. Then the following are equivalent:

1. $\hat{s}_{s,f}(U(t)) = 0$ for all $s, f$, and $t$.

2. $S = S^T$, i.e. $S$ is a Dirichlet operator.

Hence, $S$ arises from a symmetric graph with non-negative edge weights.
Quantum toy version

Another related, but less well known example comes from quantum processes. Let $H$ be Hermitian and define $U(t) = e^{-itH}$. The quantum amplitude current is defined to be $\hat{q}_{s,f}(U(t)) := U(t)_{fs} - U(t)_{sf} = 0$. Then the following are equivalent:

1. $\hat{q}_{s,f}(U(t)) = 0$ for all $s$, $f$, and $t$.
2. $H = H^T$.
3. $[H, K] = 0$, where $K$ is the antiunitary operator defined by complex conjugation in the same basis as $H$.

Hence, $H$ arises from a symmetric graph with real-valued edge weights.
These two examples are related mathematically:

1. Symmetric stochastic generators form a strict subset of symmetric matrices, so \( \hat{s}_{s,f}(U(t)) = 0 \) for all \( s, f \) and \( t \) implies that \( \hat{q}_{s,f}(U(t)) = 0 \) for all \( s, f, \) and \( t \).

2. for a quantum process with unitary operator \( e^{-itH} \), if \( H = H^T \), then \( H \) must be real and thus time symmetric.

3. So both of the aforementioned examples arise as subsets of time-reversal chiral quantum walks.

However, we shall see that there are other quantum walks which are time-reversible that do not fall into the above situations.
Gauge transformations

If $U(t) = e^{-itH}$, where $H$ is Hermitian, then for any diagonal unitary matrix $\Lambda$,

$$P_{s \rightarrow f}(\Lambda U(t) \Lambda^\dagger) = \left| \langle f | e^{-it\Lambda H \Lambda^\dagger} | s \rangle \right|^2$$

$$= \left| \langle f | e^{-itH} | s \rangle \right|^2 = P_{s \rightarrow f}(U(t))$$

(5)

We call this the $\Lambda$-action. It can be thought of as a local change of basis, i.e., a diagonal unitary

$$U_d |n\rangle = e^{i\alpha_n} |n\rangle.$$  

(6)
Gauge transformations

Figure: Effect of the gauge transformation $|n\rangle \rightarrow e^{i\alpha_n}|n\rangle$ on vertex $n$. Phases on edges can be gauge-transformed without changing the transition amplitudes, as described in the text. Here we arrange the graph as a tree rooted at $d = 0$. 

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In which network geometries do transition probabilities depend on the complex phases $\alpha_{ij}$ of the edges of the (effective Hamiltonian’s) internode coupling graph? We are interested in how the transition probabilities in the site basis depend on $\alpha_{ij}$ and if certain values of the $\alpha_{ij}$ can break time-reversal symmetry.

<table>
<thead>
<tr>
<th>Network Geometry</th>
<th>Probability time symmetric ($\forall \alpha_{ij}$)?</th>
<th>Probability depends on $\alpha_{ij}$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear chains</td>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td>trees (possibly with self-edges)</td>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td>bipartite graphs (only even cycles)</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>non-bipartite graphs (some odd cycles)</td>
<td>NO</td>
<td>YES</td>
</tr>
</tbody>
</table>

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Classification results

The support of an $n \times n$ matrix $M$ is a digraph with $n$ vertices and an edge from vertex $i$ to vertex $j$ if $M_{ij} \neq 0$.

Theorem

A Hamiltonian $H$ is conjugate to its negative under the $\Lambda$-action if and only if its support is bipartite

Theorem

A Hamiltonian $H$ is time symmetric if and only if all of its cycles have real weights or its support is bipartite with self-loops all with identical weights.
Results on disorder and $\alpha$ independence

Theorem
If $H$ is conjugate to a real matrix via $U(1)^n$, then for any real diagonal matrix $D$, $H + D$ is time symmetric.

Theorem
The probability function is independent of the choice of $\alpha_{ij}$ if and only if the underlying graph of $H$ is a tree (with possible self-loops).

Note that we have an idea to probe the relation of the effect with disorder experimentally.
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Conclusions

- Any quantum process in finite dimensions, including quantum circuits, algorithms, quantum gates, protocols and models of coherent and open quantum transport can be viewed as a single particle quantum walk on a graph.
- By changing to the single-particle framework, we have found the effect to be omnipresent—yet previously unnoticed—in a range of such quantum information protocols and algorithms.

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